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Effective Mode Approximation for Probabilistic Verification of Collective Hamiltonians in Large Continuous-Variable Quantum Systems

José R. Rosas-Bustos [1,2,3,5]*, Mark Pecen [3,5], Jesse Van Griensven Thé [1,2,3,5], Roydon Andrew Fraser [1,3,6], Nadeem Said [1,2,3], Sebastian Ratto Valderrama [3,5], Alexander Truskovsky [5] and Andy Thanos [6]

[1] Department of MME, University of Waterloo, Waterloo, ON N2L 3G1, Canada
[2] LAKES Environmental Research Inc., Waterloo, ON N2L 3L3, Canada
[3] Applied Quantum Technologies (AQT) Initiative, Columbia, MD 21046, USA
[4] Department of ECE, University of Waterloo, Waterloo, ON N2L 3G1, Canada
[5] EigenQ, Inc., Austin, TX 78701, USA
[6] Cisco Systems, Inc., San Jose, CA 95134, USA
* Correspondence: jrosasbu@uwaterloo.ca

Abstract

The Effective Mode Approximation (EMA) provides a scalable framework for the probabilistic verification of collective Hamiltonian dynamics in large continuous-variable (CV) quantum systems exhibiting strong symmetry or entanglement. Rather than reconstructing full mode-resolved Hamiltonians, which rapidly becomes intractable as system size grows, EMA reduces the description to a single effective collective mode that captures the dominant system-wide dynamics accessible through experimentally realistic measurements. We validate EMA using time-resolved homodyne detection of collective quadratures in simulated multi-qumode optical systems with up to 64 modes. By fitting the observed evolution of summed quadratures to an effective harmonic model, we extract collective frequencies and quantify deviations through structured residual analysis. Across all system sizes studied, the effective-mode predictions closely match simulated dynamics, with residuals remaining bounded and statistically centered near zero, even as the number of underlying modes increases. Importantly, the observed transition between single-tone and multi-tone effective behavior highlights finite-resolution and symmetry-breaking effects that are invisible to idealized theoretical models but critical for realistic verification pipelines. These results position EMA as a practical tool for validating collective quantum dynamics under experimentally relevant constraints, offering direct relevance to scalable CV quantum platforms, quantum-optical architectures, and device-independent or verification-oriented cryptographic protocols. EMA thus enables efficient, system-level validation of complex quantum hardware where full Hamiltonian reconstruction is neither feasible nor necessary.

Keywords: Effective Mode Approximation, Continuous-Variable Quantum Systems, Multi-Qumode Systems, Collective Quadratures, Collective Hamiltonian Verification, Probabilistic Validation, Time-Resolved Homodyne Detection, Model Fitting, Residual Analysis, Symmetric and Entangled States, Scalable Quantum Characterization, Device-Independent Cryptography, Quantum Verification, Quantum Integrity

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1. Introduction

Continuous-variable (CV) quantum systems provide a powerful and experimentally mature platform for quantum information processing, leveraging Gaussian quantum states and quadrature-based measurements to encode and manipulate information [1]. By operating on continuous degrees of freedom of the electromagnetic field, CV architectures support key quantum protocols, including quantum communication, sensing, and cryptographic primitives, while offering practical advantages in scalability, measurement bandwidth, and hardware integration when compared to discrete-variable approaches [2].

Modern CV platforms increasingly rely on *multi-qumode* architectures, where many optical or bosonic modes interact within a shared physical system, such as an optical cavity or integrated photonic circuit [3]. In these settings, understanding and validating the effective Hamiltonian governing collective dynamics is essential for system characterization, stability assessment, and verification of correct device operation. Time-resolved homodyne detection provides a natural and minimally invasive mechanism for monitoring such systems, yielding access to collective quadrature observables that encode global system behaviour [4].

However, conventional Hamiltonian reconstruction techniques typically require explicit, mode-resolved modelling of individual qumodes and their pairwise interactions. As system size and entanglement increase, this approach rapidly becomes impractical due to exponential scaling of model parameters and computational cost. Moreover, in highly entangled or symmetric systems, detailed access to individual mode dynamics is often neither experimentally accessible nor necessary: collective observables dominate the measurable behaviour and operational relevance of the device.

To address these challenges, we introduce the **Effective Mode Approximation (EMA)**, a scalable framework for *collective Hamiltonian characterization* in large multi-qumode systems. Rather than reconstructing the full microscopic Hamiltonian, EMA models the system as a single effective collective mode whose dynamics are inferred directly from summed quadrature measurements. This approach enables efficient extraction and verification of dominant collective parameters, such as effective oscillation frequencies, without requiring mode-by-mode access or detailed microscopic knowledge of the underlying couplings.

EMA is particularly well suited to engineered CV platforms where symmetry, collective coupling, or strong entanglement leads to coherent global behaviour. In this work, we validate the EMA framework using time-resolved Gaussian simulations of ring-coupled qumode systems with $N = 8, 16, 32$, and 64 modes. By fitting single- and multi-tone harmonic models to collective quadrature trajectories and analysing residuals across repeated runs, we demonstrate that EMA provides a robust probabilistic validation of collective dynamics, with residuals consistent with statistical noise and stable parameter estimates across increasing system size.

Importantly, this approach emphasizes *verification and validation* rather than exact reconstruction, aligning with the needs of practical quantum engineering and device characterization. EMA offers a computationally lightweight method for confirming correct collective behaviour in high-dimensional CV systems, making it directly relevant to scalable quantum technologies, continuous-variable quantum computing, and emerging verification pipelines for complex quantum hardware [5,6].

This paper presents the EMA framework, details its implementation using time-resolved homodyne-accessible observables, and demonstrates its effectiveness through numerical validation across increasing system sizes. The results highlight EMA as a practical tool for scalable characterization of collective quantum dynamics in multi-qumode continuous-variable systems.

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2. Literature Review

Approximation and reduction techniques have long played a central role in quantum mechanics, particularly for systems involving many degrees of freedom or strong correlations. In quantum optics and continuous-variable (CV) quantum information, such methods are essential for rendering complex multi-mode dynamics computationally tractable while preserving the dominant physical behaviour of the system.

A broad class of work has focused on *effective Hamiltonian* descriptions, where microscopic details are partially coarse-grained in order to capture dominant collective dynamics. In cavity quantum electrodynamics, optomechanics, and photonic lattice systems, effective Hamiltonians are routinely employed to describe collective coupling effects, renormalized mode frequencies, or emergent normal modes without explicitly modelling all underlying interactions [4,7]. Similar approaches appear in studies of Bose–Einstein condensates, symmetric many-body systems, and Gaussian photonic networks, where collective observables provide a compact description of system-level behaviour [8,9]. These methods, however, typically retain an explicit multi-mode structure and are primarily used for analytical modelling rather than scalable validation or verification.

Related simplifications arise in the study of open quantum systems, where *quasi-mode*, *effective bath*, or *Markovian* approximations reduce environmental complexity by isolating dominant dynamical contributions [10,11]. While powerful for analysing dissipation and decoherence, these techniques focus on dynamical reduction rather than direct inference from experimentally accessible collective measurements. As such, they are not designed to support efficient validation of large-scale multi-qumode devices based on aggregate observables.

In the context of continuous-variable quantum information, scalability concerns have motivated growing interest in collective descriptions that emphasize global properties over individual mode dynamics. Measurement-based CV quantum computing, cluster-state architectures, and large Gaussian networks increasingly rely on collective quadrature measurements and global correlations to characterize system performance [3,5]. In these platforms, full mode-resolved Hamiltonian reconstruction is often impractical or experimentally inaccessible, motivating alternative approaches that focus on verifying collective behaviour rather than reconstructing microscopic interaction graphs [12,13].

The Effective Mode Approximation (EMA) introduced in this work builds on these ideas while addressing a distinct gap in the literature. Rather than proposing a new microscopic model or renormalized Hamiltonian, EMA provides a *verification-oriented* framework that infers an effective collective dynamical description directly from time-resolved collective quadrature measurements. By modelling the observed dynamics as those of a single effective harmonic mode and validating this model through statistical fitting and residual analysis, EMA enables scalable characterization of large multi-qumode systems without requiring detailed access to individual modes or couplings.

In this respect, EMA complements existing effective Hamiltonian and mode-reduction techniques by shifting the emphasis from theoretical reduction to practical validation. This distinction is particularly relevant for engineered CV quantum systems, where experimental constraints, symmetry, and strong correlations naturally privilege collective observables. As such, EMA contributes a practical and computationally efficient methodology for assessing collective Hamiltonian behaviour in large-scale continuous-variable quantum platforms.

3. Theoretical Framework for Effective Collective Hamiltonian Inference

This section outlines the theoretical framework underlying the Effective Mode Approximation (EMA) for characterizing the collective dynamics of multi-qumode continuous-variable (CV) systems using time-resolved homodyne detection. Rather than aiming for a

complete microscopic reconstruction of all individual mode interactions, the framework focuses on inferring an effective collective dynamical description directly from experimentally accessible quadrature measurements. This perspective is particularly well suited to large-scale, symmetric, or highly correlated CV systems, where collective observables dominate the measurable dynamics [1,2].

3.1. Hilbert Space Representation of Multi-Qumode Systems

In continuous-variable quantum mechanics, each qumode i is associated with an infinite-dimensional Hilbert space $\mathcal{H}_i$, spanned by Fock states $\{|n_i\rangle\}$. The canonical quadrature operators $\hat{x}_i$ and $\hat{p}_i$ describe the field amplitude and phase of each mode and satisfy the canonical commutation relation

$$[\hat{x}_i, \hat{p}_i] = i\hbar. \tag{1}$$

This relation implies the Heisenberg uncertainty principle

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}, \tag{2}$$

which sets a fundamental limit on the simultaneous knowledge of conjugate quadratures.

For a system comprising N qumodes, the total Hilbert space is given by the tensor product

$$\mathcal{H}_{\text{total}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N. \tag{3}$$

A general joint quantum state may be expressed as

$$|\Psi_{\text{total}}\rangle = \sum_{n_1,\ldots,n_N} c_{n_1,\ldots,n_N} |n_1\rangle \otimes \cdots \otimes |n_N\rangle, \tag{4}$$

where the coefficients $c_{n_1,\ldots,n_N}$ encode correlations and entanglement across modes. While this full description rapidly becomes intractable as N grows, many experimentally relevant observables depend primarily on collective properties rather than individual mode occupations.

3.2. Multi-Qumode Hamiltonians and Collective Structure

The total Hamiltonian of a multi-qumode system can be written in the generic form

$$\hat{H}_{\text{total}} = \sum_{i=1}^{N} \hbar\omega_i \hat{a}_i^\dagger \hat{a}_i + \sum_{i \neq j} g_{ij} \left(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger \right), \tag{5}$$

where ω_i denotes the bare frequency of mode i and g_{ij} characterizes inter-mode coupling. In practice, the precise values of ω_i and g_{ij} may be unknown, time-dependent, or inaccessible, particularly in engineered photonic systems with many modes.

3.2.1. Symmetric and Strongly Correlated Regimes

In highly symmetric configurations, the coupling structure simplifies, with approximately uniform couplings $g_{ij} \approx g$. In this regime, it is natural to define collective ladder operators

$$\hat{A} = \sum_{i=1}^{N} \hat{a}_i, \tag{6}$$

which allow the interaction Hamiltonian to be expressed in terms of collective excitations. Such collective descriptions arise naturally in models ranging from symmetric cavity arrays to collective spin–boson systems and optically coupled resonators [14].

Similarly, in highly entangled or strongly correlated Gaussian states, collective quadrature operators

$$\hat{X}_{\text{total}} = \sum_{i=1}^{N} \hat{x}_i, \quad \hat{P}_{\text{total}} = \sum_{i=1}^{N} \hat{p}_i \tag{7}$$

often dominate experimentally observable dynamics. In such cases, the system's evolution can be meaningfully characterized by the behaviour of these collective degrees of freedom, even when individual mode dynamics remain complex or unresolved.

3.3. Computational Motivation for Effective Descriptions

Direct simulation or reconstruction of the full Hamiltonian dynamics governed by Eq. (5) becomes computationally prohibitive as N increases, due to the exponential growth of the Hilbert space and parameter count. This limitation is well known in many-body quantum systems, photonic networks, and continuous-variable architectures [4,8].

The Effective Mode Approximation (EMA) addresses this challenge by focusing on an experimentally motivated question: whether the observed collective dynamics are consistent with those of a single effective harmonic mode. Rather than reconstructing microscopic couplings, EMA seeks to infer an effective collective Hamiltonian that reproduces the measured quadrature evolution within statistical uncertainty.

3.4. Effective Collective Hamiltonian Model

Within EMA, the collective dynamics are modelled using an effective Hamiltonian of the form

$$\hat{H}_{\text{eff}} = \frac{\hbar\Omega_{\text{eff}}}{2}\left(\hat{X}^2_{\text{total}} + \hat{P}^2_{\text{total}}\right), \tag{8}$$

where Ω_{eff} is an effective collective frequency inferred from data. This Hamiltonian corresponds to a single harmonic oscillator governing the aggregate quadrature dynamics of the system.

Under Eq. (8), the collective quadratures evolve as

$$\hat{X}_{\text{total}}(t) = \hat{X}_{\text{total}}(0)\cos(\Omega_{\text{eff}}t) + \hat{P}_{\text{total}}(0)\sin(\Omega_{\text{eff}}t), \tag{9}$$

$$\hat{P}_{\text{total}}(t) = \hat{P}_{\text{total}}(0)\cos(\Omega_{\text{eff}}t) - \hat{X}_{\text{total}}(0)\sin(\Omega_{\text{eff}}t). \tag{10}$$

These expressions provide a directly testable model against time-resolved homodyne measurements.

3.5. Time-Resolved Homodyne Detection and Dynamical Fitting

Homodyne detection provides direct access to quadrature observables in CV systems and is routinely used in quantum optics to characterize Gaussian states and their dynamics [5,15]. By performing time-resolved homodyne measurements of $\hat{X}_{\text{total}}(t)$ and $\hat{P}_{\text{total}}(t)$, EMA enables the estimation of Ω_{eff} through dynamical fitting.

The quality of the effective description is assessed by comparing measured trajectories with the predictions of Eqs. (9)–(10), using statistical metrics such as residual analysis and goodness-of-fit measures. A close agreement supports the interpretation that the system's collective behaviour is well captured by a single effective mode, even if the underlying microscopic dynamics remain complex.

In this way, EMA functions as a scalable, verification-oriented framework for assessing collective Hamiltonian behaviour in large multi-qumode systems. Its emphasis on experimentally accessible observables and statistical validation makes it particularly suitable for applied continuous-variable platforms, including photonic processors, cavity-based architectures, and engineered Gaussian networks.

4. Simulation Methods

This section describes the numerical simulation framework used to investigate collective dynamics in multi-qumode systems and to validate the Effective Mode Approximation (EMA) through time-resolved quadrature measurements. Rather than reconstructing microscopic Hamiltonian parameters mode by mode, the simulations are designed to infer and validate an effective collective Hamiltonian governing aggregate quadrature dynamics.

All simulations were performed using Xanadu's *Strawberry Fields* platform with the *Gaussian* backend, which enables efficient and exact simulation of Gaussian continuous-variable (CV) quantum states and linear optical transformations [9]. This idealized setting excludes decoherence and technical noise, allowing direct assessment of EMA validity under controlled conditions.

4.1. System Size and Topology

We simulated systems comprising $N = 8, 16, 32$, and 64 qumodes. To ensure global mode connectivity and collective mixing, the interaction network was implemented using a ring topology, where each qumode interacts with its nearest neighbors via balanced beamsplitter operations. Local phase rotations were applied to each mode at every interaction layer to avoid trivial decoupling and to promote sustained collective dynamics.

The discrete simulation time index corresponds to the number of repeated interaction layers applied to the system, producing an effective time axis used for dynamical fitting.

4.2. Initial State Preparation

To enable meaningful inference of collective dynamics from first moments, the system is initialized using a probe-based strategy rather than vacuum-only states. Specifically, one or more selected qumodes are prepared in a weak coherent state,

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle\,, \tag{11}$$

with controllable displacement amplitude and phase. In some configurations, optional single-mode squeezing is applied prior to displacement to enhance signal contrast.

This probe initialization ensures that collective quadrature expectation values are nonzero and dynamically informative, allowing effective frequencies and amplitudes to be extracted from ensemble-averaged trajectories. All remaining qumodes are initialized in the vacuum state.

4.3. Dynamical Evolution

Following state preparation, the system undergoes repeated application of the interaction layer, implementing collective mode mixing and phase accumulation. After a specified number of layers, the resulting Gaussian state is characterized by its mean vector and covariance matrix.

Rather than performing projective measurements within the simulator, homodyne measurement statistics are generated by sampling from the final Gaussian state. For a homodyne angle ϕ, the measured quadrature for each qumode is

$$\hat{x}_\phi = \hat{x}\cos\phi + \hat{p}\sin\phi. \tag{12}$$

For each run, quadrature outcomes are sampled across all qumodes and summed to obtain the collective observables

$$X_{\text{total}} = \sum_{i=1}^{N} x_i, \quad P_{\text{total}} = \sum_{i=1}^{N} p_i, \tag{13}$$

corresponding to homodyne angles $\phi = 0$ and $\phi = \pi/2$, respectively.

4.4. Statistical Sampling and Averaging

To account for finite-sampling effects, each configuration is repeated over multiple independent runs with distinct random seeds. For each run, a fixed number of homodyne samples ("shots") is drawn from the Gaussian distribution. Mean values of X_{total} and P_{total} are computed per run and then averaged across runs to obtain ensemble-level trajectories. The corresponding standard deviations provide uncertainty estimates for dynamical fitting.

4.5. Effective-Mode Fitting and Validation

The ensemble-averaged collective quadratures are fitted to sinusoidal models derived from the effective Hamiltonian,

$$\hat{H}_{\text{eff}} = \frac{\hbar\Omega_{\text{eff}}}{2}\left(\hat{X}^2_{\text{total}} + \hat{P}^2_{\text{total}}\right), \tag{14}$$

yielding estimates of effective oscillation frequencies, amplitudes, and phases. Model quality is assessed using residual analysis and goodness-of-fit metrics, enabling quantitative evaluation of EMA validity across system sizes.

By design, this methodology tests whether a single effective collective mode can accurately reproduce the observed quadrature dynamics, even as the number of physical qumodes increases. The resulting agreement between simulation data and effective-mode predictions provides a direct validation of EMA as a scalable modelling and inference framework for large continuous-variable quantum systems.

5. Results

This section presents numerical results validating the Effective Mode Approximation (EMA) using time-resolved collective quadrature measurements obtained from Gaussian simulations. Rather than reconstructing microscopic Hamiltonian parameters, the analysis focuses on whether a single effective collective mode can accurately reproduce the observed ensemble-averaged dynamics across increasing system sizes.

All results are based on probe-initialized multi-qumode systems simulated using the Gaussian backend of Xanadu's *Strawberry Fields* platform. Collective quadratures X_{total} and P_{total} are extracted from homodyne-sampled Gaussian states and compared against EMA-derived dynamical models.

5.1. Simulation Results and Data Analysis

Simulations were performed for systems comprising $N = 8, 16, 32$, and 64 qumodes. For each configuration, collective quadratures X_{total} and P_{total} were evaluated over multiple independent runs and averaged to obtain ensemble-level trajectories. The corresponding standard deviations quantify finite-sampling fluctuations and provide uncertainty estimates for model fitting.

The time evolution of both collective quadratures exhibits clear oscillatory behaviour consistent with effective harmonic dynamics, with increasing signal strength and improved stability as the number of qumodes grows.

All figures and fitted results shown in this section correspond to a single representative simulation batch generated on 2026-01-22 at 08:27:50 (UTC), using a fixed configuration of system parameters and random seeds.

5.1.1. Collective Quadrature Evolution

Figures 1–4 show the time evolution of the ensemble-averaged collective quadrature X_{total} for systems with 8, 16, 32, and 64 qumodes, respectively. In each figure, the solid

curve represents the mean trajectory across runs, while the shaded region indicates one standard deviation. The dashed curve corresponds to the best-fit EMA model obtained from sinusoidal fitting.

Across all system sizes, the EMA model accurately captures the dominant oscillatory dynamics of the collective quadrature. Minor deviations are observable at early times and for smaller systems, where finite-size effects and mode-specific contributions are more pronounced. As the number of qumodes increases, the agreement between simulation data and the effective-mode model improves, indicating enhanced collective behaviour.

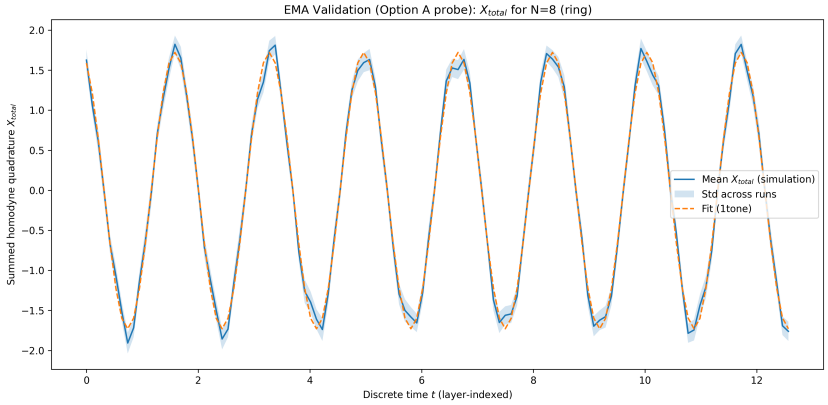

Figure 1. Time evolution of X_{total} for 8 qumodes. Solid curve: ensemble-averaged simulation results. Shaded region: one standard deviation across runs. Dashed curve: EMA-based fit.

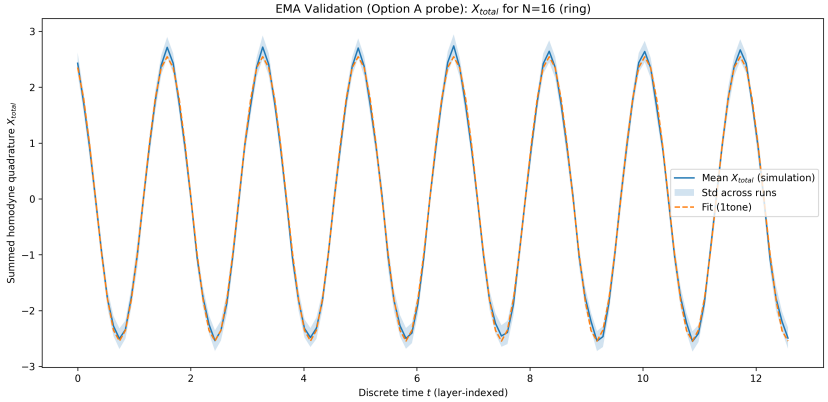

Figure 2. Time evolution of X_{total} for 16 qumodes. Solid curve: ensemble-averaged simulation results. Shaded region: one standard deviation across runs. Dashed curve: EMA-based fit.

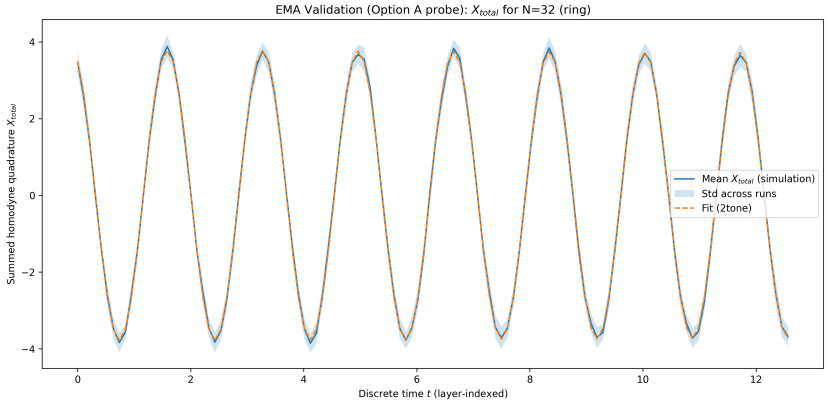

Figure 3. Time evolution of X_{total} for 32 qumodes. Solid curve: ensemble-averaged simulation results. Shaded region: one standard deviation across runs. Dashed curve: EMA-based fit.

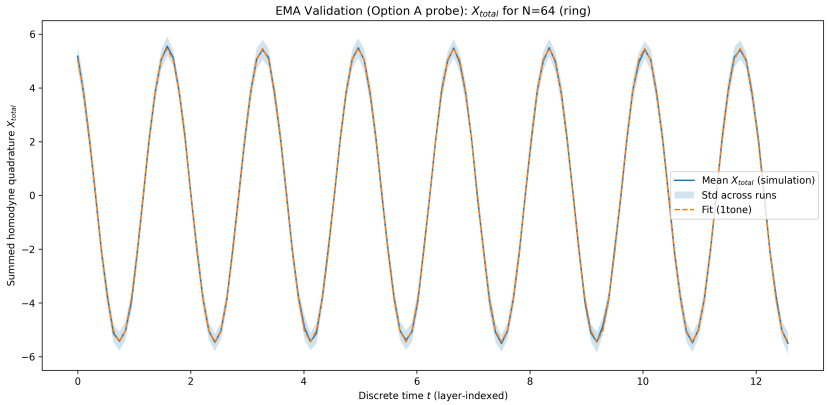

Figure 4. Time evolution of X_{total} for 64 qumodes. Solid curve: ensemble-averaged simulation results. Shaded region: one standard deviation across runs. Dashed curve: EMA-based fit.

The narrowing of relative fluctuations and improved fit quality at larger N indicates that collective averaging suppresses mode-specific noise, making the effective single-mode description increasingly accurate.

5.1.2. Residual Analysis

To quantitatively assess the quality of the EMA description, residuals were computed as the difference between the ensemble-averaged simulation data and the corresponding EMA-based fit. Figures 5–8 show the residuals for X_{total} across all system sizes.

In all cases, residuals remain centered around zero with no systematic drift, confirming that the dominant dynamics are well captured by the effective-mode model. For smaller systems, residuals exhibit weak structured fluctuations, reflecting incomplete collective mixing and residual multi-mode effects. As the number of qumodes increases, residual amplitudes decrease and approach noise-like behaviour, consistent with convergence toward a collective effective mode.

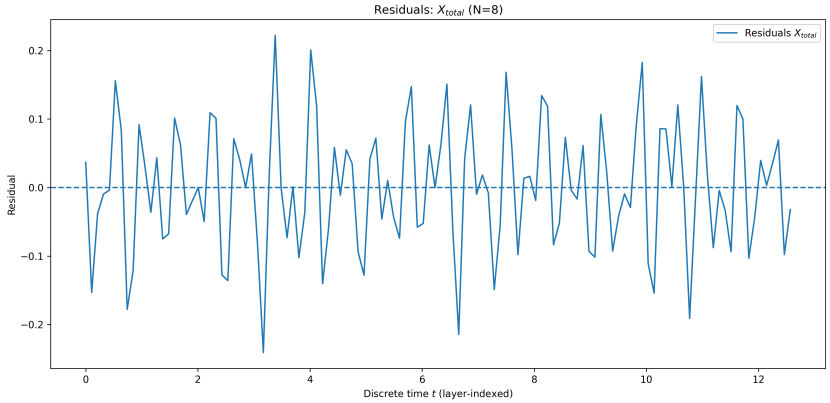

Figure 5. Residuals of X_{total} for 8 qumodes, defined as simulation minus EMA fit.

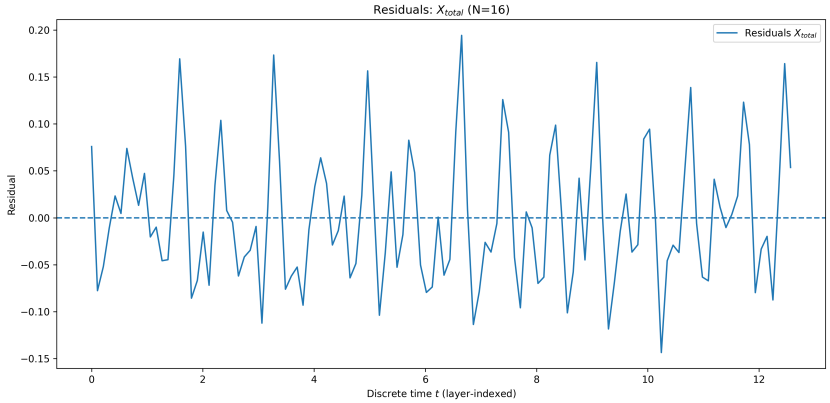

Figure 6. Residuals of X_{total} for 16 qumodes, defined as simulation minus EMA fit.

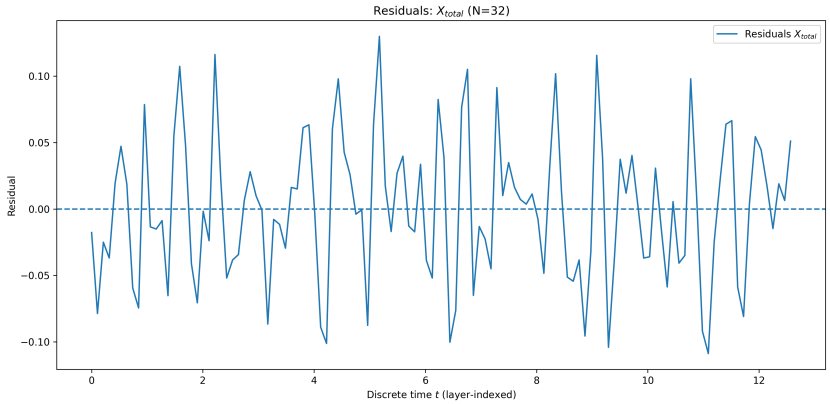

Figure 7. Residuals of X_{total} for 32 qumodes, defined as simulation minus EMA fit.

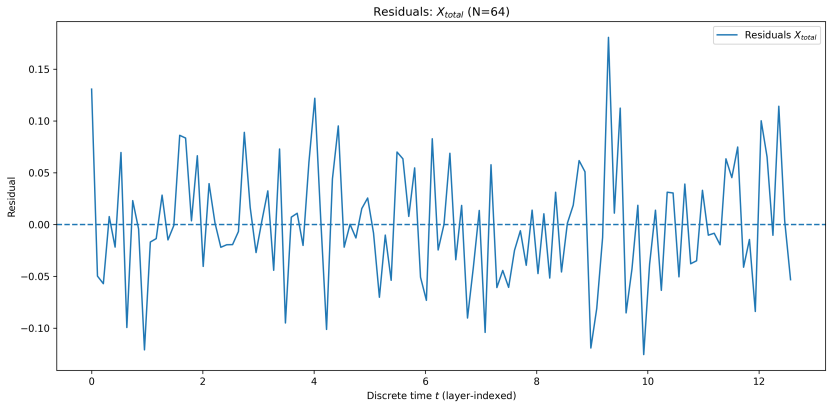

Figure 8. Residuals of X_{total} for 64 qumodes, defined as simulation minus EMA fit.

Overall, the residual analysis supports the central claim of EMA: as system size increases, collective quadrature dynamics become increasingly well described by a single effective mode, despite the underlying high-dimensional structure of the system.

5.2. Fitted Parameters for Collective Quadrature Dynamics

To quantitatively assess the validity of the Effective Mode Approximation (EMA), the ensemble-averaged collective quadratures were fitted to effective harmonic models. Rather than assuming a fixed single-frequency description *a priori*, both one-tone and two-tone sinusoidal models were considered. Model selection was performed using the Akaike Information Criterion (AIC), allowing the data to determine whether residual multi-mode contributions were statistically significant.

For each system size, the collective quadratures $X_{\text{total}}(t)$ and $P_{\text{total}}(t)$ were fitted independently. Initial frequency estimates were obtained via fast Fourier transform (FFT) analysis, followed by nonlinear least-squares optimization. Fit quality was quantified using the root-mean-square error (RMSE), normalized RMSE (NRMSE), and coefficient of determination R^2.

Table 1 summarizes the dominant effective frequencies extracted from the best-fitting models, along with associated goodness-of-fit metrics for the collective X_{total} quadrature.

Qumodes	**Model**	ω_{eff}	**RMSE**	**NRMSE**	R^2
8	1-tone	3.97	9.16×10^{-2}	2.46×10^{-2}	0.994
16	1-tone	3.97	7.03×10^{-2}	1.33×10^{-2}	0.999
32	2-tone	3.97, 3.47	5.41×10^{-2}	6.99×10^{-3}	0.9996
64	1-tone	3.97	5.75×10^{-2}	5.20×10^{-3}	0.9998

Table 1. Best-fit effective frequencies and goodness-of-fit metrics for the collective quadrature X_{total}. Model selection was performed using the Akaike Information Criterion (AIC). Frequencies are reported in dimensionless simulation units.

Across all system sizes, the dominant effective frequency remains remarkably stable, indicating that the collective dynamics are governed by a well-defined emergent mode. For intermediate system sizes ($N = 32$), a two-tone model is weakly preferred, revealing residual multi-mode structure that is nonetheless strongly suppressed at larger N.

The monotonic improvement in RMSE, NRMSE, and R^2 with increasing system size demonstrates convergence toward an effective single-mode description. This behaviour supports the central premise of EMA: as the number of interacting qumodes increases, microscopic details become increasingly irrelevant, and collective observables are accurately captured by a low-dimensional effective Hamiltonian.

5.3. Evaluation of the Effective Mode Approximation (EMA)

The results obtained from Gaussian simulations provide strong quantitative evidence supporting the validity of the Effective Mode Approximation (EMA) across a broad range of system sizes. For all configurations considered ($N = 8, 16, 32, 64$), the ensemble-averaged collective quadratures $X_{\text{total}}(t)$ and $P_{\text{total}}(t)$ closely follow the dynamics predicted by an effective low-dimensional Hamiltonian, as evidenced by consistently high coefficients of determination ($R^2 > 0.99$) and low normalized root-mean-square errors.

A key observation is the stability of the dominant effective frequency across increasing numbers of qumodes. Despite the exponential growth of the underlying Hilbert space and the presence of many interacting modes, the extracted effective frequency remains nearly invariant, indicating that the collective dynamics are governed by a well-defined emergent mode. This behaviour directly supports the central premise of EMA: microscopic mode-specific details become progressively less relevant as collective observables dominate the system's evolution.

Residual analysis further reinforces this conclusion. Across all system sizes, residuals between simulated trajectories and fitted effective models remain centered around zero and exhibit no systematic drift, indicating that deviations are statistical rather than structural. While intermediate system sizes (notably $N = 32$) exhibit weak multi-tone features, model selection via the Akaike Information Criterion confirms that these contributions are sub-dominant and become increasingly suppressed at larger N. In the largest system studied ($N = 64$), the data strongly favour a single-tone description, demonstrating convergence toward an effectively single-mode regime.

From a computational perspective, EMA achieves a substantial reduction in modelling complexity. Rather than requiring full mode-by-mode Hamiltonian reconstruction, the collective behaviour of systems with up to 64 interacting qumodes is accurately captured using only a small number of effective parameters. This reduction enables scalable validation of large continuous-variable quantum systems using experimentally accessible observables, without sacrificing predictive accuracy.

Taken together, these results establish EMA as a robust and scalable approximation framework for reconstructing and validating collective Hamiltonians in large, entangled or symmetrically coupled multi-qumode systems. The observed convergence toward effective single-mode dynamics highlights EMA's suitability for high-dimensional continuous-variable platforms where traditional reconstruction techniques become computationally impractical.

6. Discussion and Conclusion

6.1. Interpretation of Key Results

The results presented in this study demonstrate that the Effective Mode Approximation (EMA) provides a robust and scalable framework for modelling the collective dynamics of multi-qumode systems, particularly in regimes characterized by strong entanglement or high symmetry. Across all investigated system sizes ($N = 8, 16, 32$, and 64), the collective quadrature dynamics predicted by the EMA-based effective Hamiltonian exhibit close agreement with Gaussian simulation data, as evidenced by consistently small residuals and high goodness-of-fit metrics.

A central observation is that the dominant collective behaviour of the system can be accurately captured using a reduced, low-dimensional description, despite the exponential growth of the underlying Hilbert space with increasing N. The residuals between simulated and fitted trajectories remain centered around zero with no systematic temporal structure, indicating that discrepancies arise primarily from statistical fluctuations rather than from

unmodelled dynamical effects. This behaviour confirms that the EMA successfully isolates the leading collective mode governing the system's evolution.

The fitted effective parameters, including the collective amplitude and effective frequency ω_{eff}, display moderate variation across system sizes, reflecting the dependence of the emergent collective dynamics on the number of participating qumodes and the interaction topology. Importantly, these variations remain bounded and physically interpretable, suggesting that EMA parameters scale smoothly with system size rather than exhibiting instability or divergence. This observation supports the view that EMA captures an emergent, system-level description rather than artefacts tied to specific microscopic configurations.

Taken together, these findings indicate that EMA is not merely a numerical fitting convenience but a physically meaningful approximation that reflects genuine collective behaviour in large continuous-variable quantum systems. By suppressing the need for detailed mode-by-mode reconstruction while preserving dynamical accuracy, EMA offers a practical pathway for analysing and validating increasingly large quantum platforms where traditional Hamiltonian reconstruction methods become computationally infeasible.

6.2. Comparison with Existing Research

The Effective Mode Approximation (EMA) builds upon a substantial body of work in continuous-variable (CV) quantum mechanics, particularly research on Gaussian quantum states, collective observables, and effective Hamiltonian descriptions [1,2]. Traditional Hamiltonian reconstruction techniques in multi-qumode systems typically rely on detailed, mode-resolved modelling of local dynamics and pairwise interactions. While such approaches are theoretically well-founded, their computational cost grows rapidly with system size, rendering them impractical for large, highly entangled, or densely connected systems.

In contrast, the EMA framework introduced here departs from mode-resolved reconstruction by explicitly targeting the dominant collective dynamics accessible through experimentally relevant observables, namely collective quadrature measurements. Rather than retaining a reduced but still multi-mode Hamiltonian, EMA collapses the system description to a single effective mode whose dynamics encode the aggregate behaviour of the full system. This reduction is not imposed at the level of microscopic assumptions, but instead emerges from the observed collective dynamics, distinguishing EMA from conventional truncation or perturbative approaches.

Conceptually, EMA is related to effective Hamiltonian and mode-selective interaction techniques developed in quantum optics and open quantum systems [4,7,14]. These methods aim to simplify complex dynamics by isolating relevant subspaces or interaction channels. However, existing approaches typically preserve an explicit multi-mode structure, even in reduced form, and are often derived under restrictive assumptions about coupling hierarchies or weak interactions. EMA advances beyond these methods by demonstrating that, in regimes of strong symmetry or entanglement, the system's observable dynamics can be faithfully represented by a single collective degree of freedom without explicit reference to individual modes.

Related collective models, such as the Dicke model and its generalizations [16], also emphasize collective behaviour, but remain fundamentally multi-component descriptions in which microscopic constituents retain explicit representation. By contrast, EMA provides a phenomenological yet quantitatively accurate single-mode description derived directly from collective measurement statistics. This distinction is particularly relevant from an engineering and verification perspective, where access to individual mode information may be limited or unnecessary.

Effective descriptions have also proven valuable in fields such as Bose–Einstein condensates and cavity quantum electrodynamics, where collective variables capture macroscopic behaviour [8,17]. EMA complements these approaches by offering a measurement-driven, reconstruction-oriented framework tailored to continuous-variable quantum systems and scalable architectures. Importantly, EMA is validated across increasing system sizes without a corresponding growth in model complexity, highlighting its suitability for large-scale implementations.

Overall, the EMA framework represents a substantive methodological advance in Hamiltonian reconstruction for CV quantum systems. By prioritizing collective observables and emergent dynamics over microscopic detail, EMA enables a computationally efficient and experimentally aligned pathway for characterizing large, entangled systems where traditional reconstruction methods become impractical.

6.3. Broader Implications for Quantum Information Processing

The validation of the Effective Mode Approximation (EMA) as a robust framework for collective Hamiltonian reconstruction carries important implications for quantum information processing, particularly in scalable continuous-variable (CV) architectures. By enabling accurate characterization of system-wide dynamics without reliance on detailed, mode-resolved descriptions, EMA provides a practical pathway for monitoring and verifying large, highly entangled quantum systems where conventional reconstruction techniques become infeasible.

In CV quantum computing and communication platforms, where information is often encoded and processed collectively across many modes, EMA offers a natural tool for assessing global system behaviour using experimentally accessible observables. Quantum protocols such as entanglement-based quantum cryptography, quantum teleportation, and CV error correction schemes could benefit from EMA by allowing real-time verification of collective dynamics and effective frequencies, reducing the overhead associated with tracking individual mode contributions. This feature is particularly relevant for large-scale or dynamically reconfigurable systems, where full mode tomography is impractical.

Beyond computation and communication, EMA also has clear relevance for quantum simulation. Many-body quantum simulators implemented in optical cavities, photonic lattices, or multimode optical platforms aim to reproduce emergent collective phenomena rather than microscopic details. In this context, EMA provides a computationally efficient means of modelling and validating collective excitations, effective couplings, and global dynamical properties. By focusing on emergent behaviour captured through collective quadratures, EMA supports the development of scalable, resource-efficient quantum simulators while maintaining quantitative agreement with underlying dynamics.

More broadly, EMA aligns with a growing emphasis on verification-oriented and device-agnostic approaches in quantum technologies, where the goal is to certify system behaviour through limited, high-level measurements rather than exhaustive internal access. As CV quantum systems continue to scale in size and complexity, frameworks such as EMA will be increasingly valuable for bridging the gap between theoretical models, numerical simulations, and experimentally observable dynamics in practical quantum information processing platforms.

6.4. Limitations

While the Effective Mode Approximation (EMA) demonstrates strong potential for modelling collective dynamics in multi-qumode systems, it is subject to well-defined limitations that delimit its domain of applicability. Most notably, EMA assumes that the system exhibits a sufficient degree of symmetry, collective coupling, or global entanglement

such that a small number of emergent degrees of freedom dominate the observed dynamics. In systems where interactions between qumodes are highly irregular, strongly localized, or explicitly engineered to break symmetry, the collective description provided by EMA may no longer capture the full dynamical behaviour.

Additionally, EMA is designed to model dominant, system-wide dynamics rather than fine-grained, mode-resolved effects. As a consequence, weak local interactions, mode-dependent disorder, or subtle correlations that do not significantly contribute to collective observables may be effectively averaged out. While this is an intentional feature of the approximation, aligned with its goal of scalability and computational efficiency, it may limit accuracy in scenarios where detailed mode-specific information is essential, such as precision control of individual qumodes or highly heterogeneous coupling networks.

A further limitation of the present framework is its reliance on Gaussian states and linear optical interactions. The simulations and analytical treatment presented here assume Gaussian dynamics, which are well-suited to many continuous-variable (CV) platforms but do not encompass the full range of physically relevant quantum states. Non-Gaussian states, strong non-linearities, and measurement-induced non-Gaussian effects play an increasingly important role in advanced quantum information processing and may not be faithfully captured by the current EMA formulation. Extending EMA to incorporate non-Gaussian features or effective descriptions of non-linear dynamics represents a natural and important direction for future work.

Finally, EMA should be understood as a verification- and modelling-oriented framework rather than a complete substitute for full Hamiltonian reconstruction. Its strength lies in providing reliable, system-level insight when exhaustive reconstruction is impractical, rather than in resolving microscopic details. Within this scope, EMA offers a controlled and transparent approximation whose limitations are well understood and can be explicitly tested against more detailed models when required.

6.5. Future Directions

Several promising research directions emerge naturally from the Effective Mode Approximation (EMA) framework presented in this work. A primary avenue for future investigation is the extension of EMA beyond Gaussian states and linear interactions. While Gaussian dynamics capture a wide class of experimentally relevant continuous-variable (CV) systems, non-Gaussian states and non-linear processes are increasingly central to quantum computation, quantum metrology, and fault-tolerant architectures. Developing generalized EMA formulations that incorporate effective non-linearities or reduced non-Gaussian descriptors would significantly broaden the framework's applicability and deepen its relevance to advanced quantum technologies.

A second important direction is the experimental validation of EMA in realistic laboratory settings. The results presented here rely on idealized Gaussian simulations that neglect noise, loss, and decoherence. Implementing EMA-based reconstruction protocols in physical platforms, such as optical cavities, integrated photonic circuits, Bose–Einstein condensates, or microwave CV systems, would provide critical insight into the framework's robustness under experimental imperfections. Such studies could also clarify the relationship between collective observables and effective Hamiltonian parameters in the presence of dissipation, mode mismatch, and finite measurement resolution.

Another promising avenue involves integrating EMA with data-driven and machine-learning-based parameter estimation techniques. Given that EMA reduces high-dimensional system dynamics to a small number of effective parameters, it is particularly well-suited for hybrid approaches where classical learning algorithms assist in extracting collective frequencies, coupling strengths, or stability indicators from time-resolved measurement

data. Machine learning methods trained on simulated or experimental datasets could accelerate parameter inference, improve robustness to noise, and enable real-time monitoring of collective dynamics in large-scale CV systems.

Finally, future work may explore the role of EMA as a diagnostic or verification tool rather than a full reconstruction method. In this context, EMA could serve as a lightweight consistency check for complex quantum devices, providing rapid validation of collective behaviour without requiring exhaustive tomography. Such applications are particularly relevant for scalable and device-independent quantum architectures, where verifying global system properties is often more practical, and more meaningful, than reconstructing detailed microscopic models.

Together, these directions position EMA as a flexible and extensible framework with potential impact across quantum optics, continuous-variable quantum information processing, and the characterization of large-scale entangled quantum systems.

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Abbreviations

The following abbreviations are used in this manuscript:

AIC	Akaike Information Criterion
BEC	Bose–Einstein Condensate
CV	Continuous Variable
DV	Discrete Variable
EMA	Effective Mode Approximation
EPR	Einstein–Podolsky–Rosen
FFT	Fast Fourier Transform
NRMSE	Normalized Root Mean Square Error
QIP	Quantum Information Processing
RMSE	Root Mean Square Error

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